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Existence and multiplicity of solutions for asymptotically Hamiltonian elliptic systems in $\mathbb{R}^N$ ☆

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ABSTRACT

This paper is concerned with the following nonperiodic Hamiltonian elliptic system

$$\begin{cases} -\Delta u + V(x)u = H_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = -H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where the potential V is bounded below, and H is asymptotically linear in z as $|z| \to \infty$ with $z = (u,v)$. By applying a generalized linking theorem of strongly indefinite functionals, we obtain solutions for the above system.

1. Introduction and main results

In this paper we study the following nonperiodic elliptic system in Hamiltonian form:

$$\begin{cases} -\Delta u + V(x)u = H_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = -H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{$\mathcal{F}$}$$

where $N \geqslant 1$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. Set $\mathcal{J}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the system $(\mathcal{F})$ can be written as

$$-\Delta z + V(x)z = \mathcal{J}_0 H_z(x,z), \qquad z(x) = \big(u(x), v(x)\big) \to 0 \quad \text{as } |x| \to \infty,$$

which can be regarded as the stationary system of the nonlinear vector Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \gamma(x)\phi - \mathcal{J}_0 f\big(x, |\phi|\big)\phi,$$

where $\phi(x,t) = z(x)e^{-\frac{iEt}{\hbar}}$, $V(x) = \frac{2m}{\hbar^2}(\gamma(x) - E)$ and $H_z(x,z) = \frac{2m}{\hbar^2} f(x,|z|)z$.

For the case of bounded domain, many mathematicians are devoted to study of the existence and multiplicity of solutions for the system $(\mathcal{F})$. For example, see [12,13,17–19,21] and the references therein. V. Benci and P.H. Rabinowitz [17] first considered the system $(\mathcal{F})$. By using the direct minimax method, and they obtained solutions for the system $(\mathcal{F})$ in the space of $H_0^1(\Omega) \times H_0^1(\Omega)$. Later, Hulshof and van der Vorst [13] also considered the system $(\mathcal{F})$. Instead of working in

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the space of $H_0^1(\Omega) \times H_0^1(\Omega)$, the authors used a suitable family of products of fractional Sobolev spaces, and turned out that these kinds of spaces are the natural ones for this problem. By doing so, the authors also obtained solutions for the system $(\mathcal{F})$. de Figueiredo and Felmer [12] considered the superquadratic case for the system $(\mathcal{F})$. By applying a critical point theorem, they acquired solutions for the system $(\mathcal{F})$. In [21], Kryszewski and Szulkin considered the following system

$$\begin{cases} -\triangle u = F_v(x,u,v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\triangle v = F_u(x,u,v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{X}$}$$

By developing an infinite-dimensional Morse theory for strongly indefinite functionals, for $N \geqslant 1$, they obtained one weak solution for the system $(\mathcal{X})$. In particular, if $N = 1$, they proved that the system $(\mathcal{X})$ has at least two nontrivial solutions. Recently, by developing a critical point theorem for strongly indefinite functionals, de Figueiredo and Ding [18] studied the system

$$\begin{cases} -\triangle u = H_u(x,u,v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\triangle v = -H_v(x,u,v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where H satisfies $H(x,u,v) \sim |u|^p + |v|^q + R(x,u,v)$ with $\lim_{|u|+|v|\to\infty} \frac{R(x,u,v)}{|u|^p+|v|^q} = 0$, $1 < p < 2^* = \frac{2N}{N-2}$ and $q > 1$. Under some additional conditions on R, they proved the existence of multiple solutions for the above system. More recently, de Figueiredo, do Ó and Ruf [19] treated the following autonomous system via an Orlicz space approach

$$\begin{cases} -\triangle u = g(v), \quad u(x) > 0 & \text{in } \Omega, \\ -\triangle v = f(u), \quad v(x) > 0 & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Under some superlinear conditions on f and g, the authors obtained one solution for the system.

Up till now, some authors considered the system $(\mathcal{F})$ in the space of $\mathbb{R}^N$. See, e.g., [6,14,15,20,22–27] and the references therein. Most of them focused on the case $V \equiv 1$. The main difficulty of this problem is the lack of compactness for the Sobolev's embedding theorem. In the early results [14,15,20,22], the authors managed to recover the compactness for the Sobolev's embedding, by working in the space of radially symmetric functions. In this way, de Figueiredo and Yang [14] considered the system

$$\begin{cases} -\triangle u + u = g(x,v) & \text{in } \mathbb{R}^N, \\ -\triangle v + v = f(x,u) & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{Y}$}$$

They proved that the system $(\mathcal{Y})$ has a radial solution under the assumptions that $f(x,t)$ and $g(x,t)$ are superlinear in t and radially symmetric with respect to x, $|f(x,t)| \leqslant c(1+t^{p-1})$ and $|g(x,t)| \leqslant c(1+t^{q-1})$ with $2 \leqslant p,q \leqslant \frac{2N}{N-2}$. Later, their results were generalized by Sirakov [20] to the system

$$\begin{cases} -\triangle u + k(x)u = g(x,v) & \text{in } \mathbb{R}^N, \\ -\triangle v + k(x)v = f(x,u) & \text{in } \mathbb{R}^N. \end{cases}$$

In [22], by using the fountain theorem, Bartsch and de Figueiredo proved that the system $(\mathcal{F})$ admits infinitely many radial as well as non-radial solutions. Recently, by using generalized linking theorem, Li and Yang [15] considered the following system

$$\begin{cases} -\triangle u + V(x)u = R_v(x,u,v) & \text{in } \mathbb{R}^N, \\ -\triangle v + V(x)v = R_u(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{$\mathcal{Z}$}$$

They proved that there exists a positive ground state solution for the system $(\mathcal{Z})$ with asymptotically linear nonlinearities. Here, a solution is called a ground state if $(u,v) \neq 0$ and its energy is minimal among the energy of all the nontrivial bound state of $(\mathcal{Z})$. Moreover, they also proved the existence of a positive solution for $(\mathcal{Z})$, if $f(x,u) \to \bar{f}(x)$ and $g(x,u) \to \bar{g}(x)$ as $|u| \to \infty$.

In the papers [23–26], by using the dual variational method, the authors obtained solutions for the system $(\mathcal{Z})$. Later, in [5], Bartsch and Ding developed a generalized linking theorem for the strongly indefinite functionals (see [4,16] for related results), which provided another way to deal with such problems. By applying the theorems of [5], Zhao and Ding [6] considered the periodic asymptotically linear system $(\mathcal{Z})$, and they first obtained the multiple solutions for the system $(\mathcal{Z})$ under the stronger assumptions. Recently, in the paper [27], we considered nonperiodic superquadratic case for the system $(\mathcal{Z})$. By applying the generalized linking theorem of [5], the authors proved that the system $(\mathcal{Z})$ has at least one solution. As far as we know, for nonperiodic asymptotically linear case, there is still no result. The main purpose of this paper is to consider such case. The main difficulty of this problem is the lack of compactness of the Sobolev's embedding theorem. Moreover, the variational functional is strongly indefinite. In such case, it is difficulty for us to get the compactness for the Cerami-sequences. In order to overcome this difficulty, we manage to recover sufficient compactness by imposing a control on the size of $H(x,z)$ with respect to the behavior of $V(x)$ at infinity in x (see the following condition (H_3)). This approach was originally developed in Jeanjean [29] (see also [1]).

Set $A := -\Delta + V(x)$. Denote by $\sigma(A)$, $\sigma_d(A)$ and $\sigma_e(A)$ the spectrum, the point spectrum and the essential spectrum of the operator A, respectively. Assume that V and H satisfy the following conditions:

($\mathcal{V}_0$) $V(x) \in C(\mathbb{R}^N, [0,\infty))$, and there exists some $K > 0$ such that $\Theta_K := \{x \in \mathbb{R}^N\colon\ V(x) < K\}$ is nonempty and has finite measure;

(H_0) $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$ and $H_z(x,z) = o(z)$ as $|z| \to 0$ uniformly in x;

(H_1) $H(x,z) = \frac{1}{2}F(x)z\cdot z + R(x,z)$, where $F(x) \in L^\infty(\mathbb{R}^N)$ and $\frac{R_z(x,z)}{|z|} \to 0$ as $|z| \to \infty$;

(H_2) $F_0 := \inf_{x\in\mathbb{R}^N} F(x) > \underline{\lambda} := \inf[\sigma(A) \cap (0,+\infty)]$;

(H_3) $H^* := \limsup_{|x|\to\infty} \sup_{z\neq 0} \frac{|H_z(x,z)|}{|z|} < K_0$, where $K_0 := \sup\{K\colon\ |\Theta_K| < \infty\}$;

(H_4) One of the following holds:
 (i) $0 \notin \sigma(\mathcal{J}_0 A - F(x))$;
 (ii) $\tilde{H}(x,z) \geqslant 0$ and $\tilde{H}(x,z) \geqslant \alpha_0$ for some $\alpha_0 > 0$ and all x with $|z|$ large enough, where $\tilde{H}(x,z) := \frac{1}{2}H_z(x,z)z - H(x,z)$.

From the definitions of F_0 and H^*, we have $F_0 \leqslant H^* < K_0$. Set $\ell := \sharp\{\sigma(A) \cap (0,F_0)\}$. Our main result is the following:

Theorem 1.1. *Let ($\mathcal{V}_0$), (H_0)–(H_4) be satisfied. Then the system ($\mathcal{F}$) has at least one solution. If, in addition, $H(x,z)$ is even in z, then the system ($\mathcal{F}$) has at least ℓ pairs of solutions.*

Remark 1.1. The following function satisfies (H_0)–(H_2) and (H_4);

*Ex*1. $H(x,z) := b(x)|z|^2(1 - \frac{1}{\ln^2(e+|z|)})$, where $0 < \inf b(x)$ and $b(x) \in C(\mathbb{R}^N, \mathbb{R})$.

Form now on, the letter c will be indiscriminately used to denote various constants whose exact values are irrelevant.

2. Variational setting

In this section, we shall establish variational framework for the system ($\mathcal{F}$). For the convenience of notation, let $|\cdot|_q$ denote the usual L^q-norm and $(\cdot,\cdot)_2$ be the usual $\mathcal{H} := L^2(\mathbb{R}^N)$-inner product. In order to prove Theorem 1.1, we introduce the space

$$E_+ := \left\{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)u^2 < \infty\right\}$$

which is a Hilbert space equipped with the inner product

$$(u_1,u_2)_+ := \int_{\mathbb{R}^N} \big(\nabla u_1 \nabla u_2 + V(x)u_1u_2\big)$$

and the associated norm $\|u\|_+^2 = (u,u)_+$. By the assumption ($\mathcal{V}_0$), it is easy to check that E_+ embeds continuously in $H^1(\mathbb{R}^N)$. Let $\lambda_e := \inf \sigma_e(A)$, then we have the following lemma.

Lemma 2.1. *Suppose ($\mathcal{V}_0$) holds. Then $\lambda_e \geqslant K_0$.*

Proof. We follow the ideal of Lemma 2.4 in [31]. Let $K > 0$ such that $|\Theta_K| < \infty$. Set $W(x) = V(x) - K$, $W^\pm := \max\{\pm W(x), 0\}$ and $\mathcal{A} := -\triangle + K + W^+$. It follows from ($\mathcal{V}_0$) that the multiplicity operator W^- is compact relative to $\mathcal{A}$ (cf. [32]). By the Weyl's theorem, we know that

$$\sigma_e(A) \subset \sigma_e(\mathcal{A}) \subset [K,+\infty),$$

as required. □

Remark 2.1. Noting that it is possible that $\lambda_e = \infty$, i.e., $\sigma_d(A) = \sigma(A)$. For example, this is the case if $V(x) \to \infty$ as $|x| \to \infty$.

From now on, we fix a number K with

$$H^* < K < K_0, \tag{2.1}$$

where H^* is defined in (H_3). Let k be the number of the eigenvalues less than K. We write η_i and h_i ($1 \leqslant i \leqslant k$) for the eigenvalues and eigenfunctions, respectively. Setting

$$\mathcal{H}^d := \{h_1,\ldots,h_k\},$$

we have the orthogonal decomposition

$$\mathcal{H} := \mathcal{H}^d \oplus \mathcal{H}^e, \quad u = u^d + u^e.$$

Correspondingly, E_+ has the decomposition:

$$E_+ := E_+^d \oplus E_+^e \quad \text{with } E_+^d = \mathcal{H}^d \text{ and } E_+^e = E_+ \cap \mathcal{H}^e,$$

orthogonal with respect to both inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)_+$. By Lemma 2.1, we have that

$$K|u|_2^2 \leqslant \|u\|_+^2, \quad \forall u \in E_+^e. \tag{2.2}$$

Let S be the best Sobolev constants such that $S|u|_{2^*}^2 \leqslant \int_{\mathbb{R}^N} |\nabla u|^2$. It is clear that

$$S|u|_{2^*}^2 \leqslant \|u\|_+^2, \quad \forall u \in E_+.$$

It follows from E_+ embeds continuously in $H^1(\mathbb{R}^N)$ that

$$c|u|_2^2 \leqslant \|u\|_+^2, \quad \forall u \in E_+.$$

Together with interpolation, it follows that for each $s \in [2, 2^*]$

$$c|u|_s^s \leqslant \|u\|_+^s, \quad \forall u \in E_+. \tag{2.3}$$

In the sequel, we write $|z| = (|u|^2 + |v|^2)^{\frac{1}{2}}$ for $z = (u, v)$. Let

$$E := E^+ \oplus E^-,$$

where

$$E^+ := E_+ \times \{0\} \quad \text{and} \quad E^- := \{0\} \times E_+.$$

For $z = (u, v) \in E$, we write $z^+ := (u, 0) \in E^+$ and $z^- := (0, v) \in E^-$. Obviously, E is a Hilbert space under the inner product

$$(z_1, z_2) := (u_1, u_2)_+ + (v_1, v_2)_+$$

and the induced norm

$$\|z\|^2 := \|u\|_+^2 + \|v\|_+^2.$$

Set $E^e := E_+^e \times E_+^e$ and $E^d := E_+^d \times E_+^d$. Then E also has the following orthogonal decomposition

$$E = E^e \oplus E^d.$$

Accordingly, we write $z = z^e + z^d$ for $z \in E$ with $z^d = (u^d, v^d)$ and $z^e = (u^e, v^e)$. It follows from (2.2) that

$$K|z|_2^2 \leqslant \|z\|^2, \quad \forall z \in E^e. \tag{2.4}$$

Moreover, by (2.3), we know that for all $s \in [2, 2^*]$

$$c|z|_s^2 \leqslant \|z\|^2, \quad \forall z \in E. \tag{2.5}$$

Thus, we have the following lemma.

Lemma 2.2. *Assume $(\mathcal{V}_0)$ is satisfied. Then E embeds continuously into $L^p(\mathbb{R}^N, \mathbb{R}^2)$ $(\forall p \in [2, 2^*])$. Moreover, the embedding $E \hookrightarrow L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ $(\forall p \in [2, 2^*))$ is compact.*

Obviously, the system $(\mathcal{F})$ has the following energy functional

$$\Phi(z) = \Phi(u, v) = \int_{\mathbb{R}^N} \frac{1}{2}\big(|\nabla u|^2 + V(x)u^2 - |\nabla v|^2 - V(x)v^2\big) - \Psi(z), \quad \text{for } z = (u, v) \in E,$$

where $\Psi(z) = \int_{\mathbb{R}^N} H_z(x, z)$. From the above space decomposition, it follows that the functional Φ can be rewritten in a standard way

$$\Phi(z) = \Phi(u, v) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|v\|^2 - \Psi(z),$$

for all $z = (u, v) = z^+ + z^- \in E$. By assumptions and Lemma 2.2, we know that $\Psi \in C^1(E, \mathbb{R})$ and $\Phi \in C^1(E, \mathbb{R})$. Moreover, a standard argument shows that the critical points of Φ are solutions of $(\mathcal{F})$ (see, e.g., [6,18]).

3. The abstract critical point theorem

In order to study the critical points of Φ, we now recall some abstract critical point theory developed recently in [3,5]. Also see [4,16] for earlier related results.

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space with direct sum decomposition $\mathbb{E} = X \oplus Y$, and P_X, P_Y denote the projections onto X, Y, respectively. For a functional $\Phi \in C^1(\mathbb{E}, \mathbb{R})$, we write $\Phi_a := \{z \in \mathbb{E}: \ \Phi(z) \geqslant a\}$, $\Phi^b := \{z \in \mathbb{E}: \ \Phi(z) \leqslant b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Next, let's us recall some definitions:

(i) Φ is said to be weakly sequentially upper semi-continuous, i.e., $z_n \rightharpoonup z$ in $\mathbb{E}$ one has $\Phi(z) \geqslant \liminf_{n\to\infty} \Phi(z_n)$;
(ii) Φ' is said to be weakly sequentially continuous, i.e., $\lim_{n\to\infty} \Phi'(z_n)w = \Phi'(z)w$ for each $w \in \mathbb{E}$;
(iii) A sequence $\{z_n\} \subset \mathbb{E}$ is said to be a $(C)_c$-sequence if $\Phi(z_n) \to c$ and $(1 + \|z_n\|)\Phi'(z_n) \to 0$. Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

From now on we assume that the Banach space X is separable and reflexive, and fix a countable dense subset $B \subset X^*$. For each $b \in B$ we define a semi-norm on $\mathbb{E}$ by

$$P_b : \mathbb{E} = X \oplus Y \to \mathbb{R}, \quad P_b(x + y) = q_b(x) + \|y\|, \quad \text{for } x + y \in X \oplus Y,$$

where $q_b(x) = |(x, b)_{X,X^*}| = |b(x)|$. We denote by $\mathcal{T}_B$ the induced topology. Let w^* denote the *weak**-topology on $\mathbb{E}^*$.

Assume:

$(\mathcal{K}_0)$ For any $c > 0$, there exists $\xi > 0$ such that $\|z\| < \xi\|P_Y z\|$ for all $z \in \Phi_c$.
$(\mathcal{K}_1)$ For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_B$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_B) \to (\mathbb{E}^*, w^*)$ is continuous.
$(\mathcal{K}_2)$ There exists $\varrho > 0$ with $\kappa := \inf \Phi(S_\varrho Y) > 0$ where $S_\varrho Y := \{z \in Y: \ \|z\| = \varrho\}$.

Then we have the following theorem.

Theorem 3.1. *Let $(\mathcal{K}_0)$–$(\mathcal{K}_2)$ be satisfied and suppose there are $R > \varrho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi|_{\partial M} \leqslant \kappa$ where $M := \{z = x + se; \ s \geqslant 0, \ x \in X, \ \|z\| \leqslant R\}$. Then Φ has a $(C)_c$-sequence with $\kappa \leqslant \Phi(z) \leqslant \tilde{c} := \sup \Phi|_M$. Moreover, if Φ satisfies the $(C)_c$-condition for all $c \leqslant \tilde{c}$, then Φ has a critical point z with $\kappa \leqslant \Phi(z) \leqslant \tilde{c}$.*

This theorem is a special case of Theorem 4.4 in [1]. For stating the multiple critical points theorem for the functional Φ, we further assume:

$(\mathcal{K}_3)$ There are a finite-dimensional subspace $Y_0 \subset Y$ and $R > \varrho$ such that for $E_0 := X \oplus Y_0$ and $B_0 := \{u \in E_0; \ \|u\| \leqslant R\}$, we have $\tilde{c} := \sup \Phi|_{E_0} < \infty$ and $\sup \Phi|_{E_0 \setminus B_0} < \inf \Phi|_{B_\varrho \cap Y}$, where $B_\varrho = \{z \in E: \ \|z\| \leqslant \varrho\}$.

The following theorem is a special case of Theorem 4.6 in [5].

Theorem 3.2. *If Φ is even, and satisfies $(\mathcal{K}_1)$–$(\mathcal{K}_3)$ and the $(C)_c$-condition for all $c \in [\kappa, \tilde{c}]$, then it has at least $\mathfrak{M} := \dim Y_0$ pairs of critical points with critical values less or equal to $\tilde{c}$.*

4. The linking structure

Recalling that we denote $|z| = (|u|^2 + |v|^2)^{1/2}$ for $z = (u, v) \in E$. In this section we study the linking structure for the functional Φ. Clearly, under the assumptions (H_0)–(H_4), we know that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$H_z(x, z) \leqslant \varepsilon|z| + C_\varepsilon|z|^{p-1} \tag{4.1}$$

and

$$H(x, z) \leqslant \varepsilon|z|^2 + C_\varepsilon|z|^p \tag{4.2}$$

for (x, z), where $p \in [2, 2^*)$. First we have the following lemma.

Lemma 4.1. *Suppose $(\mathcal{V}_0)$, (H_0)–(H_4) are satisfied. Then there is a $\varrho > 0$ such that $\kappa := \inf \Phi|_{\partial B_\varrho \cap E^+} > 0$.*

Proof. For $z^+ = (u, 0) \in E^+$, by (4.2) and Lemma 2.2 we have

$$\begin{aligned}\Phi(z^+) &= \frac{1}{2}\|z^+\|^2 - \int_{\mathbb{R}^N} H(x, z^+) \geqslant \frac{1}{2}\|z^+\|^2 - \varepsilon|z^+|_2^2 - C_\varepsilon|z^+|_p^p \\ &\geqslant \left(\frac{1}{2} - c\varepsilon\right)\|z^+\|^2 - cC_\varepsilon\|z^+\|^p.\end{aligned}$$

Let $\varepsilon < \frac{1}{4c}$, we know that the lemma holds with $\|z^+\|$ small enough. $\square$

In the following, we arrange all the eigenvalues (counted with multiplicity) of A in $(0, F_0)$ by $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_\ell < F_0$ and let β_i denote the corresponding eigenfunctions: $A\beta_i = \lambda_i \beta_i$ for $i = 1, \ldots, \ell$. Set $\mathcal{W}_0 := span\{\beta_1, \ldots, \beta_\ell\}$. Obviously,

$$\lambda_1 |w|^2 \leqslant \|w\|^2 \leqslant \lambda_\ell |w|^2, \quad \forall w \in \mathcal{W}_0. \tag{4.3}$$

For any finite-dimensional subspace $\mathcal{W}$ of $\mathcal{W}_0$, set $E_{\mathcal{W}} := E^- \oplus \mathcal{W}$.

Lemma 4.2. *Under the assumptions of Lemma 4.1, we have that for any subspace $\mathcal{W}$ of $\mathcal{W}_0$, $\sup \Phi|_{E_{\mathcal{W}}} < \infty$. Moreover, there exists $R_{\mathcal{W}} > 0$ such that $\Phi(z) < \inf \Phi|_{B_\varrho \cap E^+}$ for all $z \in E_{\mathcal{W}}$ with $\|z\| \geqslant R_{\mathcal{W}}$, where $\varrho > 0$ is given in Lemma 4.1.*

Proof. It suffices to prove that $\Phi(z) \to -\infty$ in $E_{\mathcal{W}}$ as $\|z\| \to \infty$. If not, then there are $Q > 0$ and $\{z_j\} \subset E_{\mathcal{W}}$ with $\|z_j\| \to \infty$ such that $\Phi(z_j) \geqslant -Q$ for all j, where $z_j = (u_j, v_j)$. Denote $y_j := \frac{z_j}{\|z_j\|}$, passing to a subsequence if necessary, $y_j \rightharpoonup y$, $y_j^- \rightharpoonup y^-$ and $y_j^+ \to y^+$. Since $H(x,z) \geqslant 0$, then we have

$$\begin{aligned}\frac{1}{2}\left(\|y_j^+\|^2 - \|y_j^-\|^2\right) &\geqslant \frac{1}{2}\left(\|y_j^+\|^2 - \|y_j^-\|^2\right) - \int_{\mathbb{R}^N} \frac{H(x,z_j)}{\|z_j\|^2} \\ &= \frac{\Phi(z_j)}{\|z_j\|^2} \geqslant \frac{-Q}{\|z_j\|^2},\end{aligned} \tag{4.4}$$

which yields that

$$\frac{1}{2}\|y_j^-\|^2 \leqslant \frac{1}{2}\|y_j^+\|^2 + \frac{Q}{\|z_j\|^2}. \tag{4.5}$$

We claim that $y^+ \neq 0$. Indeed, if not, it follows from (4.5) that $\|y_j^-\| \to 0$. Thus $\|y_j\| \to 0$, which contradicts with $\|y_j\| = 1$. By (4.3) one has

$$\begin{aligned}\|y^+\|^2 - \|y^-\|^2 - \int_{\mathbb{R}^N} F(x)|y|^2 &\leqslant \|y^+\|^2 - \|y^-\|^2 - F_0|y|_2^2 \\ &\leqslant -(F_0 - \lambda_\ell)\left|y^+\right|_2^2 < 0,\end{aligned}$$

then there exists $R > 0$ such that

$$\|y^+\|^2 - \|y^-\|^2 - \int_{B_R} F(x)|y|^2 < 0, \tag{4.6}$$

where $B_R := \{x \in \mathbb{R}^N\colon\ |x| \leqslant R\}$. It follows from (H_1) that $|R(x,z)| \leqslant c|z|^2$ for some $c > 0$, and $\frac{R(x,z)}{|z|^2} \to 0$ as $|z| \to \infty$ uniformly in x. Note that

$$\lim_{j\to\infty} \int_{B_R} \frac{R(x,z_j)}{\|z_j\|^2} = \lim_{j\to\infty} \int_{B_R} \frac{R(x,z_j)}{|z_j|^2}|y_j|^2 = 0. \tag{4.7}$$

Thus (4.4)–(4.6) imply that

$$\begin{aligned}0 &\leqslant \lim_{j\to\infty}\left[\frac{1}{2}\left(\|y_j^+\|^2 - \|y_j^-\|^2\right) - \int_{B_R} \frac{H(x,z_j)}{\|z_j\|^2}\right] \\ &\leqslant \frac{1}{2}\left(\|y^+\|^2 - \|y^-\|^2 - \int_{B_R} F(x)|y|^2\right) \\ &< 0.\end{aligned}$$

Now the desired conclusion follows from this contradiction. $\square$

Especially, we have the following lemma.

Lemma 4.3. *Let the assumptions of Lemma 4.2 be satisfied. Then letting $e \in \mathcal{W}_0$ with $\|e\| = 1$, there is $R_1 > 0$ such that $\Phi|_{\partial M} \leqslant \kappa$, where $M := \{z = z^- + se\colon\ z^- \in E^-,\ s \geqslant 0,\ \|z\| \leqslant R_1\}$, and κ is defined in Lemma 4.1.*

5. The $(C)_c$-sequence

In this section we shall prove that Φ satisfies the $(C)_c$-condition.

Lemma 5.1. *Let $(\mathcal{V}_0)$ and (H_0)–(H_4) be satisfied. Then any $(C)_c$-sequence is bounded.*

Proof. Let $\{z_j\} \subset E$ be such that

$$\Phi(z_j) \to c \quad \text{and} \quad \big(1 + \|z_j\|\big)\Phi'(z_j) \to 0. \tag{5.1}$$

Then, for $C_0 > 0$, one has

$$C_0 \geqslant \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}^N} \tilde{H}(x, z_j). \tag{5.2}$$

To prove that $\{z_j\}$ is bounded, we develop a contradiction argument related to the one introduced in [8] (see also [1,7]). We assume that, up to a subsequence, $\|z_j\| \to \infty$ as $j \to \infty$. Set $y_j := \frac{z_j}{\|z_j\|}$. Then $\|y_j\| = 1$. Without loss of generality, we can assume that $y_j \rightharpoonup y := (\xi, \zeta)$ in E, $y_j \to y$ in L^p_{loc} for $p \in [2, 2^*)$, and $y_j(x) \to y(x)$ for a.e. $x \in \mathbb{R}^N$. There are only two cases needed to be considered: $y \equiv 0$ or $y \neq 0$.

If $y \equiv 0$, then $y_j \rightharpoonup 0$ in E and $y_j \to 0$ in $L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ for $p \in [2, 2^*)$. Moreover, $y^d_j \to 0$ in E and $y^d_j \to 0$ in L^p_{loc}. Choosing some number K such that $H^* < K < K_0$. It follows from (H_3) that for fixed $\epsilon > 0$, there exists $R_\epsilon > 0$ such that $|H_z(x,z)| \leqslant (K - \epsilon)|z|$ for $|z| \geqslant R_\epsilon$ uniformly in x. Set $A_{\epsilon,j} = \{x \in \mathbb{R}^N\colon\ |z_j(x)| \leqslant R_\epsilon\}$ and $A^c_{\epsilon,j} = \mathbb{R}^N \setminus A_{\epsilon,j}$. It follows from

$$o(1) = \frac{\Phi'(z_j)(z^{e+}_j - z^{e-}_j)}{\|z_j\|^2} = \|y^e_j\|^2 - \int_{\mathbb{R}^N} \frac{H_z(x, z_j)}{|z_j|}\big(y^{e+}_j - y^{e-}_j\big)|y_j|$$

that

$$\begin{aligned}
\|y^e_j\|^2 &= \int_{A_{\epsilon,j}} \frac{H_z(x, z_j)}{|z_j|}\big(y^{e+}_j - y^{e-}_j\big)|y_j| + \int_{A^c_{\epsilon,j}} \frac{H_z(x, z_j)}{|z_j|}\big(y^{e+}_j - y^{e-}_j\big)|y_j| + o(1) \\
&\leqslant c\int_{A_{\epsilon,j}} \big|y^{e+}_j - y^{e-}_j\big||y_j| + (K - \epsilon)\int_{A^c_{\epsilon,j}} \big|y^{e+}_j - y^{e-}_j\big||y_j| + o(1) \\
&\leqslant (K - \epsilon)\big|y^e_j\big|^2_2 + o(1).
\end{aligned} \tag{5.3}$$

By (2.2) one gets

$$\left(1 - \frac{K - \epsilon}{K}\right)\|y^e_j\|^2 \leqslant o(1),$$

which implies that $\|y^e_j\|^2 \to 0$. Hence $1 = \|y_j\|^2 = \|y^e_j\|^2 + \|y^d_j\|^2 \to 0$, a contradiction. Therefore, $y \equiv 0$ cannot occur.

Now suppose $y \neq 0$. For each $\varphi = (\psi, \phi) \in C^\infty_0(\mathbb{R}^N, \mathbb{R}^2)$, we have

$$\begin{aligned}
\Phi'(z_j)\varphi &= \big(z^+_j - z^-_j, \varphi\big) - \int_{\mathbb{R}^N} H_z(x, z_j)\varphi \\
&= \big(z^+_j - z^-_j, \varphi\big) - \int_{\mathbb{R}^N} R_z(x, z_j)\varphi - \int_{\mathbb{R}^N} F(x)z_j\varphi \\
&= \|z_j\|\left[\big(y^+_j - y^-_j, \varphi\big) - \int_{\mathbb{R}^N} \frac{R_z(x, z_j)\varphi}{|z_j|}|y_j| - \int_{\mathbb{R}^N} F(x)y_j\varphi\right],
\end{aligned}$$

which yields that

$$o(1) = \big(y^+_j - y^-_j, \varphi\big) - \int_{\mathbb{R}^N} \frac{R_z(x, z_j)\varphi}{|z_j|}|y_j| - \int_{\mathbb{R}^N} F(x)y_j\varphi.$$

Letting $j \to \infty$, we have $(y^+ - y^-, \varphi) - (F(x)y, \varphi)_2 = 0$, i.e.,

$$\begin{cases} -\triangle\xi + V(x)\xi = F(x)\xi & \text{in } \mathbb{R}^N, \\ -\triangle\zeta + V(x)\zeta = -F(x)\zeta & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{J}$}$$

Recalling that $\mathcal{J}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the system $(\mathcal{J})$ can be rewritten as

$$-\Delta y + V(x)y = \mathcal{J}_0 F(x)y.$$

This is impossible if (i) of (H_4) is satisfied. Now we assume (ii) of (H_4) holds. We adopt an argument borrowing from Ding and Jeanjean [1]. Let $\omega := \frac{\xi+\zeta}{2}$ and $\psi := \frac{\xi-\zeta}{2}$. Then the system $(\mathcal{J})$ can be written as

$$\begin{cases} -\triangle\omega + V(x)\omega = F(x)\psi & \text{in } \mathbb{R}^N, \\ -\triangle\psi + V(x)\psi = F(x)\omega & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{I}_0$}$$

Since $y = (\xi, \zeta) \neq 0$, then $\chi = (\omega, \psi) \neq 0$. Thus, either $\omega \neq \psi$ or $\omega \equiv \psi \neq 0$ holds.

(I) If $\omega \neq \psi$, by $(\mathcal{I}_0)$ one has

$$-\Delta\phi + V(x)\phi = -F(x)\phi, \quad \text{for } \phi := \omega - \psi \neq 0. \tag{5.4}$$

(II) If $\omega \equiv \psi \neq 0$, it follows from $(\mathcal{I}_0)$ that

$$-\Delta\omega + V(x)\omega = F(x)\omega, \quad \text{for } \omega = \psi \neq 0. \tag{5.5}$$

From the condition (ii) of (H_4), then there exists some $\alpha > 0$ such that $\tilde{H}(x,z) \geqslant \alpha_0$ whenever $|z| \geqslant \alpha$. Clearly, one has

$$\begin{aligned} C &\geqslant \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}^N} \tilde{H}(x, z_j) \\ &\geqslant \int_{|z_j|\geqslant\alpha} \tilde{H}(x, z_j) \geqslant \int_{|z_j|\geqslant\alpha} \alpha_0 = \alpha_0 \bigl|\bigl\{x \in \mathbb{R}^N \colon |z_j| \geqslant \alpha\bigr\}\bigr|, \end{aligned}$$

hence

$$\bigl|\bigl\{x \in \mathbb{R}^N \colon \bigl|z_j(x)\bigr| \geqslant \alpha\bigr\}\bigr| \leqslant \frac{C}{\alpha_0}.$$

(i) If (I) holds, set $\Xi := \{x \in \mathbb{R}^N \colon \phi(x) = \omega(x) - \psi(x) \neq 0\}$. By (5.4) and the unique continuation arguments similar to those of [2,9,28,30], we deduce that $|\Xi| = \infty$. Hence there exists $\varepsilon > 0$ and $\Xi' \subset \mathbb{R}^N$ such that $|y(x)| = |(\omega+\psi, \omega-\psi)| \geqslant 2\varepsilon$ for $x \in \Xi'$ and $\frac{2C}{\alpha_0} \leqslant |\Xi'| < \infty$. By Egoroff's theorem, we can find a set $\Xi'' \subset \Xi'$ with $|\Xi''| > \frac{C}{\alpha_0}$ such that $y_j \to y$ uniformly on Ξ''. So for almost all j, $|y_j(x)| \geqslant \varepsilon$ and $|z_j| \geqslant \alpha$ in Ξ''. Then

$$\frac{C}{\alpha_0} < \bigl|\Xi''\bigr| \leqslant \bigl|\bigl\{x \in \mathbb{R}^N \colon \bigl|z_j(x)\bigr| \geqslant \alpha\bigr\}\bigr| \leqslant \frac{C}{\alpha_0}. \tag{5.6}$$

A contradiction.

(ii) If (II) holds, let $\Xi := \{x \in \mathbb{R}^N \colon \omega(x) = \psi(x) \neq 0\}$. Repeating the argument of (i), we can acquire the contradiction inequality (5.6). Thus, the sequence $\{z_j\}$ is bounded. □

Let $\{z_j\} \subset E$ be an arbitrary $(C)_c$-sequence. By Lemma 5.1 it is bounded, hence, we may assume without loss of generality that $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^q_{loc} for $q \in [2, 2^*)$ and $z_j(x) \to z(x)$ a.e. on $\mathbb{R}^N$. Obviously, z is a critical point of Φ.

Lemma 5.2. *Let $s \in [2, 2^*)$. There is a subsequence such that for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that*

$$\limsup_{n\to\infty} \int_{B_n \setminus B_r} |z_{j_n}|^s \leqslant \epsilon \tag{5.7}$$

for all $r \geqslant R_\epsilon$, where $B_k := \{x \in \mathbb{R}^N;\ |x| \leqslant k\}$.

Proof. We follow the ideal of [1]. Note that, for each $n \in \mathbb{N}$, $\int_{B_n} |z_j|^s \to \int_{B_n} |z|^s$ as $j \to \infty$, there exists $\hat{j}_n \in \mathbb{N}$ such that

$$\int_{B_n} \bigl(|z_j|^s - |z|^s\bigr)\, dt < \frac{1}{n} \quad \text{for all } j = \hat{j}_n + i,\ i = 1, 2, \ldots.$$

Without loss of generality we can assume $\hat{j}_{n+1} \geqslant \hat{j}_n$. In particular, for $j_n = \hat{j}_n + n$ we have

$$\int_{B_n} \bigl(|z_{j_n}|^s - |z|^s\bigr) < \frac{1}{n}.$$

Clearly, there exists R_ϵ satisfying

$$\int_{\mathbb{R}^N\setminus B_r} |z|^s < \epsilon, \quad \forall r \geqslant R_\epsilon. \tag{5.8}$$

Since

$$\begin{aligned}\int_{B_n\setminus B_r} |z_{j_n}|^s &= \int_{B_n} \left(|z_{j_n}|^s - |z|^s\right) + \int_{B_n\setminus B_r} |z|^s + \int_{B_r} \left(|z|^s - |z_{j_n}|^s\right) \\ &\leqslant \frac{1}{n} + \int_{\mathbb{R}^N\setminus B_r} |z|^s + \int_{B_r} \left(|z|^s - |z_{j_n}|^s\right),\end{aligned}$$

by (5.8), and the lemma now follows easily. □

In the following, we adopt a cut-off technique developed in Ackermann [10,11]. Let $\eta : [0,\infty] \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leqslant 1$, $\eta(t) = 0$ if $t \geqslant 2$. Define $\tilde{z}_j(x) = \eta(\frac{2|x|}{j})z(x)$. Clearly,

$$\|\tilde{z}_j - z\| \to 0, \quad \text{as } j \to \infty. \tag{5.9}$$

In a similar way to Lemma 3.2 in Ackermann [10] (see also Lemma 4.4 in [1]), we can obtain the following:

Lemma 5.3. *Under the assumptions of Theorem* 1.1, *we have*

$$\sup_{\|\varphi\|\leqslant 1} \left| \int_{\mathbb{R}^N} \big(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n)\big)\varphi \right| \to 0.$$

Proof. Note that (5.9) and the local compactness of Sobolev embedding imply that, for any $r > 0$,

$$\lim_{n\to\infty} \left| \int_{B_r} \big(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n)\big)\varphi \right| \to 0$$

uniformly in $\|\varphi\| \leqslant 1$. For any $\epsilon > 0$, we infer from (5.8) and (5.9) that

$$\lim_{n\to\infty} \int_{B_n\setminus B_r} |\tilde{z}_{j_n}|^s \leqslant \int_{\mathbb{R}^N\setminus B_r} |z|^s \leqslant \epsilon, \quad \text{for all } r \geqslant R_\epsilon.$$

Using (5.7) for $s = 2$, p we have

$$\begin{aligned}&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} \big(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n)\big)\varphi \right| \\ &\quad = \limsup_{n\to\infty} \left| \int_{B_n\setminus B_r} \big(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n)\big)\varphi \right| \\ &\quad \leqslant C_1 \limsup_{n\to\infty} \int_{B_n\setminus B_r} \big(|z_{j_n}| + |\tilde{z}_n|\big)|\varphi| + C_2 \limsup_{n\to\infty} \int_{B_n\setminus B_r} \big(|z_{j_n}|^{p-1} + |\tilde{z}_n|^{p-1}\big)|\varphi| \\ &\quad \leqslant C_1 \limsup_{n\to\infty} \big(|z_{j_n}|_{\mathcal{H}(B_n\setminus B_r)} + |\tilde{z}_n|_{\mathcal{H}(B_n\setminus B_r)}\big)|\varphi|_2 + C_2 \limsup_{n\to\infty} \big(|z_{j_n}|^{p-1}_{L^p(B_n\setminus B_r)} + |\tilde{z}_n|^{p-1}_{L^p(B_n\setminus B_r)}\big)|\varphi|_p \\ &\quad \leqslant C_3\epsilon^{\frac{1}{2}} + C_4\epsilon^{\frac{p-1}{p}}\end{aligned}$$

uniformly in $\|\varphi\| \leqslant 1$, and this proves the lemma. □

Lemma 5.4. *Let* $(\mathcal{V}_0)$, (H_0)–(H_4) *be satisfied. Then* Φ *satisfies the* $(C)_c$*-condition.*

Proof. Let $\{z_j\}$ be a $(C)_c$-sequence of Φ. Now we use the decomposition $E = E^e \oplus E^d$. Recall that $\dim E^d < \infty$. Set $y_n := z_{j_n} - \tilde{z}_n = y_n^e + y_n^d$. Then $y_n \rightharpoonup 0$ and $y_n^d = (z_{j_n}^d - z^d) + (z^d - \tilde{z}_n^d) \to 0$. Moreover, it follows from Lemma 5.3 that for $\varphi \in E$

$$\Phi'(y_n)\varphi = \Phi'(z_{j_n} - \tilde{z}_n)\varphi = \Phi'(z_{j_n})\varphi - \Phi'(\tilde{z}_n)\varphi + \int\limits_{\mathbb{R}^N} \big(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n)\big)\varphi.$$

Letting $n \to \infty$, one has $\Phi'(y_n) \to 0$. Set $\hat{y}_n^e := y_n^{e+} - y_n^{e-}$. Observe that

$$o(1) = \Phi'(y_n)\hat{y}_n^e = \|y_n^e\|^2 - \int\limits_{\mathbb{R}^N} H_z(x, y_n)\hat{y}_n^e.$$

Thus, similar to (5.3), we have

$$\begin{aligned}
\|y_j^e\|^2 &= \int\limits_{A_{\epsilon,j}} \frac{H_z(x, y_j)}{|y_j|}(\hat{y}_n^e)|y_j| + \int\limits_{A_{\epsilon,j}^c} \frac{H_z(x, y_j)}{|y_j|}(\hat{y}_n^e)|y_j| + o(1) \\
&\leqslant c \int\limits_{A_{\epsilon,j}} |\hat{y}_n^e||y_j| + (K - \epsilon) \int\limits_{A_{\epsilon,j}^c} |\hat{y}_n^e||y_j| + o(1) \\
&\leqslant (K - \epsilon)|y_j^e|_2^2 + o(1) \leqslant \frac{K - \epsilon}{K}\|y_j^e\|_2^2 + o(1).
\end{aligned}$$

Hence $\|y_j^e\|_2^2 \to 0$, and so $\|y_n\| \to 0$. Since $z_{j_n} - z = y_n + (\tilde{z}_n - z)$, hence $\|z_{j_n} - z\| \to 0$. This ends the proof. □

6. Proof of Theorem 1.1

In order to apply the abstract Theorems 3.1–3.2 to Φ, we choose $X = E^-$ and $Y = E^+$. X is separable and reflexive and let B be a countable dense subset of X^*. First we have

Lemma 6.1. *Φ satisfies $(\mathcal{K}_0)$.*

Proof. For any $c > 0$ and $z \in \Phi_c$, using the fact that $H \geqslant 0$ one has

$$0 < c \leqslant \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big).$$

This yields $\|z^-\| < \|z^+\|$, and hence $\|z\| < 2\|z^+\|$. □

Lemma 6.2. *Φ satisfies $(\mathcal{K}_1)$.*

Proof. We first show that Φ_a is $\mathcal{T}_B$-closed for every $a \in \mathbb{R}$. Consider a sequence $\{z_n\} \subset \Phi_a$ which $\mathcal{T}_B$-converges to $z \in E$, and write $z_n = z_n^+ + z_n^-$, $z = z^- + z^+$. Then $z_n^+ \to z^+$ in norm topology and hence $\{z_n^+\}$ is bounded in norm topology. Observe that there exists $C > 0$ such that

$$C \geqslant \|z_n^+\|^2 - 2\Phi(z_n) - \int\limits_{\mathbb{R}^N} H(x, z_n) = \|z_n^-\|^2.$$

This implies the boundedness of $\{z_n^-\}$, and hence $z_n^- \rightharpoonup z^-$. Therefore, we have $z_n \rightharpoonup z$. Since Ψ is weakly sequentially lower semi-continuous, it follows that

$$a \leqslant \lim_n \Phi(z_n) \leqslant \Phi(z),$$

thus $z \in \Phi_a$ and Φ_a is $\mathcal{T}_B$-closed.

Next we show that $\Phi' : (\Phi_c, \mathcal{T}_B) \to (\mathbb{E}^*, w^*)$ is continuous. It suffices to show that Ψ' has the same property. Suppose $z_n \rightharpoonup z$ in E. Then $z_n \to z$ in L_{loc}^p for $p \in [2, 2^*)$. It is obvious that

$$\Psi'(z_n)\varphi = \int\limits_{\mathbb{R}^N} H_z(x, z_n)\varphi \to \int\limits_{\mathbb{R}^N} H_z(x, z)\varphi = \Psi'(z)\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, as $n \to \infty$. Now using the density of $C_0^\infty(\mathbb{R}^N)$ in E we can obtain the desired conclusion. □

Proof of Theorem 1.1. *Existence*: By Lemmas 6.1–6.2, we know that the conditions $(\mathcal{K}_0)$–$(\mathcal{K}_1)$ are satisfied. Lemma 4.1 implies $(\mathcal{K}_2)$. Lemma 4.3 shows that Φ possesses the linking structure, and Lemma 5.4 implies that Φ satisfies the $(C)_c$-condition. Therefore, by Theorem 3.1, we have that Φ has at least one critical point z with $0 < \kappa \leqslant \Phi(z)$.

Multiplicity: Since $H(x, z)$ is even in z, we know that Φ is even. Lemma 4.2 says that Φ satisfies $(\mathcal{K}_3)$ with $\dim \mathcal{W}_0 = \ell$. Therefore, Φ has at least ℓ pairs of nontrivial critical points by Theorem 3.2. □

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References

[1] Y.H. Ding, L. Jeanjean, Homoclinic orbits for non periodic Hamiltonian system, J. Differential Equations 237 (2007) 473–490.
[2] Y.H. Ding, A. Szulkin, Bound states for semilinear Schrödinger equations with sign-changing potential, Calc. Var. Partial Differential Equations 29 (2007) 397–419.
[3] Y.H. Ding, Variational Methods for Strongly Indefinite Problems, World Scientific Press, 2008.
[4] T. Bartsch, Y.H. Ding, On a nonlinear Schrödinger equations, Math. Ann. 313 (1999) 15–37.
[5] T. Bartsch, Y.H. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory, Math. Nachr. 279 (2006) 1267–1288.
[6] F.K. Zhao, L.G. Zhao, Y.H. Ding, Multiple solutions for asymptotically linear elliptic systems, NoDEA Nonlinear Differential Equations Appl. 15 (2008) 673–688.
[7] F.K. Zhao, Y.H. Ding, On a diffusion system with bounded potential, Discrete Contin. Dyn. Syst. 23 (2009) 1073–1086.
[8] A. Szulkin, W. Zou, Homoclinic orbits for asymptotically linear Hamiltonian systems, J. Funct. Anal. 187 (2001) 25–41.
[9] H.P. Heinz, On the number of solutions of nonlinear Schrödinger equations and on unique continuation, J. Differential Equations 116 (1995) 149–171.
[10] N. Ackermann, On a periodic Schrödinger equation with nonlinear superlinear part, Math. Z. 248 (2004) 423–443.
[11] N. Ackermann, A nonlinear superposition principle and multibump solutions of periodic Schröinger equations, J. Funct. Anal. 234 (2006) 423–443.
[12] D.G. de Figueiredo, P.L. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1994) 99–116.
[13] J. Hulshof, R.C.A.M. van der Vorst, Differential systems with strongly variational structure, J. Funct. Anal. 114 (1993) 32–58.
[14] D.G. de Figueiredo, J. Yang, Decay, symmetry and existence of solutions of semilinear elliptic systems, Nonlinear Anal. 33 (1998) 211–234.
[15] G. Li, J. Yang, Asymptotically linear elliptic systems, Comm. Partial Differential Equations 29 (2004) 925–954.
[16] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to semi-linear Schrödinger equation, Adv. Differential Equations 3 (1998) 441–472.
[17] V. Benci, P.H. Rabinowitz, Critical point theorems for indefinite functionals, Invent. Math. 52 (1979) 241–273.
[18] D.G. de Figueiredo, Y.H. Ding, Strongly indefinite functionals and multiple solutions of elliptic systems, Trans. Amer. Math. Soc. 355 (2003) 2973–2989.
[19] D.G. de Figueiredo, J.M. do Ó, B. Ruf, An Orlicz-space approach to superlinear elliptic systems, J. Funct. Anal. 224 (2005) 471–496.
[20] B. Sirakov, On the existence of solutions of Hamiltonian elliptic systems in $\mathbb{R}^N$, Adv. Differential Equations 5 (2000) 1445–1464.
[21] W. Kryszewski, A. Szulkin, An infinite dimensional Morse theory with applications, Trans. Amer. Math. Soc. 349 (1997) 3181–3234.
[22] T. Bartsch, D.G. de Figueiredo, Infinitely many solutions of nonlinear elliptic systems, in: Progr. Nonlinear Differential Equations Appl., vol. 35, Birkhäuser, Basel, Switzerland, 1999, pp. 51–67.
[23] A.I. Ávila, J. Yang, On the existence and shape of least energy solutions for some elliptic systems, J. Differential Equations 191 (2003) 348–376.
[24] A.I. Ávila, J. Yang, Multiple solutions of nonlinear elliptic systems, NoDEA Nonlinear Differential Equations Appl. 12 (2005) 459–479.
[25] C.O. Alves, P.C. Carrião, O.H. Miyagaki, On the existence of positive solutions of a perturbed Hamiltonian system in $\mathbb{R}^N$, J. Math. Anal. Appl. 276 (2002) 673–690.
[26] J. Yang, Nontrivial solutions of semilinear elliptic systems in $\mathbb{R}^N$, Electron. J. Differential Equations 6 (2001) 343–357.
[27] J. Wang, J.X. Xu, F.B. Zhang, Existence of solutions for nonperiodic superquadratic Hamiltonian elliptic systems, Nonlinear Anal. 72 (2010) 1949–1960.
[28] Z.L. Liu, J.B. Su, Z.Q. Wang, Solutions of elliptic problems with nonlinearities of linear growth, Calc. Var. 35 (2009) 463–480.
[29] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesmann–Lazer type problem set on $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999) 787–809.
[30] F.K. Zhao, L.G. Zhao, Y.H. Ding, Existence and multiplicity of solutions for a nonperiodic Schrödinger equation, Nonlinear Anal. 69 (2008) 3671–3678.
[31] Y.H. Ding, J.C. Wei, Semiclassical states for nonlinear Schrödinger equations with sign-changing potentials, J. Funct. Anal. 251 (2007) 546–572.
[32] T. Bartsch, A. Pankov, Z.Q. Wang, Nonlinear Schrödinger equations with steep potential well, Commun. Contemp. Math. 4 (2001) 549–569.